Dilatonic quantum multi-brane-worlds

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A five-dimensional dilatonic gravity action with surface counterterms motivated by AdS/CFT (conformal fluid theory) correspondence and with contributions of brane-quantum CFTs is considered around an AdS-like bulk. The effective equations of motion are constructed. They admit two (outer and inner), or multi-brane, solutions where the brane CFTs may be different. The role of quantum-brane CFT is in inducing a complicated brane dilatonic gravity. For exponential bulk potentials the number of AdS-like bulk spaces is found in analytical form. The corresponding flat or curved (de Sitter or hyperbolic) dilatonic two branes are created, as a rule, thanks to quantum effects. The observable early universe may correspond to an inflationary brane. The found dilatonic quantum two-brane-worlds usually contain a naked singularity but in a couple explicit examples the curvature is finite and a horizon (corresponding to wormholelike space) appears.

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I. INTRODUCTION

The recent booming activity in the study of brane-worlds is caused by several reasons. First, gravity on a four-dimensional (4D) brane embedded in a higher-dimensional AdS-like universe may be localized [1,2]. Second, a way to resolve the mass-hierarchy problem appears [1]. Third, new ideas on the solution of the cosmological constant problem have come to the game [3,4]. A very incomplete list of references [5,6] (and references therein), mainly on the cosmological aspects of brane-worlds, is growing every day.

The essential element of brane-world models is the presence in the theory of two free parameters (bulk cosmological constant and brane tension, or brane cosmological constant). The role of the brane cosmological constant is to fix the position of the brane in terms of tension (that is why the brane cosmological constant and brane tension are almost the same thing). Being completely consistent and mathematically reasonable, such a way of doing things may not look completely satisfactory. Indeed, the physical origin (and prediction) of brane tension in terms of some dynamical mechanism may be required.

The ideology may be different, in the spirit of Refs. [8,7]. One considers the addition of surface counterterms to the original action on AdS-like space. These terms are responsible for making the variational procedure well-defined (in Gibbons-Hawking spirit) and for elimination of the leading divergences of the action. Brane tension is not considered as a free parameter anymore but is fixed by the condition of finiteness of space-time when brane goes to infinity. Of course, leaving the theory in such form would rule out the possibility of consistent brane-world solutions' existence. Fortunately, other parameters contribute to brane tension. If one considers that there is quantum conformal fluid theory (CFT) living on the brane (which is more close to the spirit of AdS/CFT correspondence [9]) then such CFT produces conformal anomaly (or anomaly-induced effective action). This contributes to brane tension. As a result, dynamical mechanism to get brane-world with flat or curved (de Sitter or anti-de Sitter) brane appears. The curvature of such a 4D universe is expressed in terms of some-dimensional parameter l that usually appears in AdS/CFT setup and in terms of content of quantum-brane matter. In other words, brane-world is the consequence of the fact (verified experimentally by everyday life) of the presence of matter on the brane. For example, sign of conformal anomaly terms for usual matter is such that in one-brane case the de Sitter (ever expanding, inflationary) universe is preferable solution of brane equation.[1]

The scenario of Refs. [8,7] may be extended to the presence of dilaton(s) as it was done in Ref. [12] or to formulation of quantum cosmology in Wheeler–De Witt form [13]. Then whole scenario looks even more related with AdS/CFT correspondence as dilatonic gravity naturally follows as bosonic sector of 5D gauged supergravity. Moreover, the extra prize-in form of dynamical determination of 4D boundary value of dilaton appears. In Ref. [12] the quantum dilatonic one-brane universe has been presented with the possibility to get inflationary or hyperbolic or flat brane with dynamical determination of brane dilaton. The interesting question is

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[1]Similar mechanisms for anomaly-driven inflation in a usual 4D world has been invented by Starobinsky [10] and generalized for presence of dilaton in Ref. [11].

related with generalization of such a scenario in dilatonic gravity for the multibrane case. This will be the purpose of the present work.

In the next section we present general action of 5D dilatonic gravity with surface counterterms and quantum-brane CFT contribution. This action is convenient for description of brane-worlds where bulk is AdS-like space-time. There could be one or two (flat or curved) branes in the theory. As it was already mentioned, the brane tension is fixed in our approach; instead of it the effective brane tension is induced by quantum effects. Section III is devoted to formulation of effective bulk-brane field equations. The explicit analytical solution of bulk equations for a number of exponential bulk potentials is presented. The lengthy analysis of 4D brane equations shows the possibility to have two (inner and outer) branes associated with each of the above bulk solutions. It is interesting that quantum-created branes can be flat, or de Sitter (inflationary) or hyperbolic. The role of quantum-brane-matter corrections in getting such branes is extremely important. Nevertheless, there are few particular cases where such branes appear on classical level, i.e., without quantum corrections. In Sec. IV we briefly describe how to get generalization of above solutions for quantum dilatonic multi-brane-worlds with more than two branes. A brief summary of results is given in Sec. V where also the study of character of singularities for proposed two-brane solutions is presented. In most cases, as usually occurs in AdS-dilatonic gravity, the solutions contain the naked singularity. However, in a couple cases the scalar curvature is finite and there is a horizon. The corresponding 4D branes may be interpreted as wormholes.

II. DILATONIC GRAVITY ACTION WITH BRANE-QUANTUM CORRECTIONS

Let us present the initial action for dilatonic-AdS gravity under consideration. The metric of (Euclidean) AdS has the following form:

$$ds^2=dz^2+\sum_{i,j=1}^{4} g_{(4)ij}dx^i dx^j, \quad g_{(4)ij}=e^{2\tilde{A}(z)}\hat{g}_{ij}. \tag{1}$$

Here $\hat{g}_{ij}$ is the metric of the Einstein manifold, which is defined by $r_{ij}=k\hat{g}_{ij}$, where r_{ij} is the Ricci tensor constructed with $\hat{g}_{ij}$ and k is a constant. One can consider two copies of the regions given by $z<z_0$ and glue two regions putting a brane at $z=z_0$. More generally, one can consider two copies of regions $\tilde{z}_0<z<z_0$ and glue the regions putting two branes at $z=\tilde{z}_0$ and $z=z_0$. Hereafter we call the brane at $z=\tilde{z}_0$ as ''inner'' brane and that at $z=z_0$ as ''outer'' brane.

Let us first consider the case with only one brane at $z=z_0$ and start with Euclidean signature action S that is the sum of the Einstein-Hilbert action $S_{\rm EH}$ with kinetic term and potential $V(\phi)=12/l^2+\Phi(\phi)$ for dilaton ϕ, the Gibbons-Hawking surface term $S_{\rm GH}$, the surface counterterm S_1, and the trace-anomaly-induced action[2] W:

$$S=S_{\rm EH}+S_{\rm GH}+2S_1+W, \tag{2}$$

$$S_{\rm EH}=\frac{1}{16\pi G}\int d^5x\sqrt{g_{(5)}}\left(R_{(5)}-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi+\frac{12}{l^2}+\Phi(\phi)\right), \tag{3}$$

$$S_{\rm GH}=\frac{1}{8\pi G}\int d^4x\sqrt{g_{(4)}}\nabla_\mu n^\mu, \tag{4}$$

$$S_1=-\frac{1}{16\pi Gl}\int d^4x\sqrt{g_{(4)}}\left(\frac{6}{l}+\frac{l}{4}\Phi(\phi)\right), \tag{5}$$

$$\begin{aligned}W={}&b\int d^4x\sqrt{\tilde{g}}\tilde{F}A+b'\int d^4x\sqrt{\tilde{g}}\left\{A\left[2\tilde{\Box}^2+\tilde{R}_{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu-\frac{4}{3}\tilde{R}\tilde{\Box}^2+\frac{2}{3}(\tilde{\nabla}^\mu\tilde{R})\tilde{\nabla}_\mu\right]A+\left(\tilde{G}-\frac{2}{3}\tilde{\Box}\tilde{R}\right)A\right\}\\&-\frac{1}{12}\left\{b''+\frac{2}{3}(b+b')\right\}\int d^4x\sqrt{\tilde{g}}[\tilde{R}-6\tilde{\Box}A-6(\tilde{\nabla}_\mu A)(\tilde{\nabla}^\mu A)]^2+C\int d^4x\sqrt{\tilde{g}}A\phi\\&\times\left[\tilde{\Box}^2+2\tilde{R}_{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu-\frac{2}{3}\tilde{R}\tilde{\Box}^2+\frac{1}{3}(\tilde{\nabla}^\mu\tilde{R})\tilde{\nabla}_\mu\right]\phi.\end{aligned} \tag{6}$$

Here the quantities in the five-dimensional bulk space-time are specified by the subscripts (5) and those in the boundary four-dimensional space-time are specified by (4). The factor 2 in front of S_1 in Eq. (2) is coming from the fact that we have two bulk regions that are connected with each other by the brane. In Eq. (4), n^μ is the unit vector normal to the boundary. In Eqs. (4), (5), and (6), one chooses the four-dimensional boundary metric as

$$g_{(4)\mu\nu}=e^{2A}\tilde{g}_{\mu\nu}. \tag{7}$$

We should distinguish A and $\tilde{g}_{\mu\nu}$ with $\tilde{A}(z)$ and $\hat{g}_{ij}$ in Eq. (1). We will specify $\hat{g}_{ij}$ later in Eq. (27). We also specify the quantities given by $\tilde{g}_{\mu\nu}$ by using tildes: G ($\tilde{G}$) and F ($\tilde{F}$) are

[2]For the introduction to anomaly-induced effective action in curved space-time (with torsion), see Sec. 5.5 in [14]. This anomaly-induced action is due to brane CFT living on the boundary of dilatonic AdS-like space.

the Gauss-Bonnet invariant and the square of the Weyl tensor, which are given as[3]

$$G=R^2-4R_{ij}R^{ij}+R_{ijkl}R^{ijkl},$$

$$F=\frac{1}{3}R^2-2R_{ij}R^{ij}+R_{ijkl}R^{ijkl}. \quad (8)$$

In the effective action (6) induced by brane-quantum matter, with N scalar, $N_{1/2}$ spinor, N_1 vector fields, N_2 ($=0$ or 1) gravitons, and $N_{\rm HD}$ higher-derivative conformal scalars, b, b', and b'' are

$$b=\frac{N+6N_{1/2}+12N_1+611N_2-8N_{\rm HD}}{120(4\pi)^2},$$

$$b'=-\frac{N+11N_{1/2}+62N_1+1411N_2-28N_{\rm HD}}{360(4\pi)^2},$$

$$b''=0. \quad (9)$$

Usually, b'' may be changed by the finite renormalization of local counterterm in gravitational effective action. As it was the case in Ref. [12], the term proportional to $\{b''+\frac{2}{3}(b+b')\}$ in Eq. (6), and therefore b'', does not contribute to the equations of motion. Note that CFT-matter-induced effective action may be considered as brane-dilatonic gravity.

For typical examples motivated by AdS/CFT correspondence [9] one has

(a) $\mathcal{N}=4$ $SU(N)$ super Yang-Mills (SYM) theory

$$b=-b'=\frac{C}{4}=\frac{N^2-1}{4(4\pi)^2}, \quad (10)$$

(b) $\mathcal{N}=2$ $Sp(N)$ theory

$$b=\frac{12N^2+18N-2}{24(4\pi)^2}, \quad b'=-\frac{12N^2+12N-1}{24(4\pi)^2}. \quad (11)$$

One can write the corresponding expression for dilaton-coupled spinor matter [15] that also has nontrivial (slightly different in form) dilatonic contribution to conformal anomaly (CA) than in case of holographic conformal anomaly [16] for $\mathcal{N}=4$ super Yang-Mills theory.

Let us consider the case where there are two branes at $z=\tilde{z}_0$ and $z=z_0$, adding the action corresponding to the brane at $z=\tilde{z}_0$ to the action in Eq. (2):

$$S_{\rm two\ branes}=S+\tilde{S}_{\rm GH}+2\tilde{S}_1+\tilde{W}, \quad (12)$$

$$\tilde{S}_{\rm GH}=\frac{1}{8\pi G}\int d^4x\sqrt{g_{(4)}}\nabla_\mu n^\mu, \quad (13)$$

$$\tilde{S}_1=\frac{1}{16\pi Gl}\int d^4x\sqrt{g_{(4)}}\left(\frac{6}{l}+\frac{l}{4}\Phi(\phi)\right), \quad (14)$$

$$\begin{aligned}\tilde{W}=&\tilde{b}\int d^4x\sqrt{\tilde{g}}\tilde{F}A+\tilde{b}'\int d^4x\sqrt{\tilde{g}}\\ &\times\left\{A\left[2\tilde{\Box}^2+\tilde{R}_{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu-\frac{4}{3}\tilde{R}\tilde{\Box}^2\right.\right.\\ &\left.\left.+\frac{2}{3}(\tilde{\nabla}^\mu\tilde{R})\tilde{\nabla}_\mu\right]A+\left(\tilde{G}-\frac{2}{3}\tilde{\Box}\tilde{R}\right)A\right\}\\ &-\frac{1}{12}\left\{\tilde{b}''+\frac{2}{3}(\tilde{b}+\tilde{b}')\right\}\int d^4x\sqrt{\tilde{g}}\\ &\times[\tilde{R}-6\tilde{\Box}A-6(\tilde{\nabla}_\mu A)(\tilde{\nabla}^\mu A)]^2\\ &+\tilde{C}\int d^4x\sqrt{\tilde{g}}A\phi\left[\tilde{\Box}^2+2\tilde{R}_{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\right.\\ &\left.-\frac{2}{3}\tilde{R}\tilde{\Box}^2+\frac{1}{3}(\tilde{\nabla}^\mu\tilde{R})\tilde{\nabla}_\mu\right]\phi. \end{aligned} \quad (15)$$

We should note that the relative sign of $\tilde{S}_1$ is different from S_1. The parameters $\tilde{b}$, $\tilde{b}'$, $\tilde{b}''$, and $\tilde{C}$ correspond to the matter that may be different from the outer brane one on the inner brane as in Eq. (9). Hence, the situation with different CFTs on the branes may be considered. Having the action at hand one can study its dynamics.

III. DILATONIC QUANTUM-BRANE-WORLDS

Let us start the consideration of field equations for the two-branes model. First of all, one defines a new coordinate z by

$$z=\int dy\sqrt{f(y)}, \quad (16)$$

and solves y with respect to z. Then the warp factor is $e^{2\hat{A}(z,k)}=y(z)$. Here one assumes the metric of five-dimensional space-time as follows:

$$ds^2=g_{(5)\mu\nu}dx^\mu dx^\nu=f(y)dy^2+y\sum_{i,j=1}^{4}\hat{g}_{ij}(x^k)dx^i dx^j. \quad (17)$$

Here $\hat{g}_{ij}$ is the metric of the four-dimensional Einstein manifold as in Eq. (1). From the variation over $g_{(5)\mu\nu}$ in the Einstein-Hilbert action (3), we obtain the following equation in the bulk:

[3]We use the following curvature conventions:

$$R=g^{\mu\nu}R_{\mu\nu},$$

$$R_{\mu\nu}=R^\lambda{}_{\mu\lambda\nu},$$

$$R^\lambda{}_{\mu\rho\nu}=-\Gamma^\lambda_{\mu\rho,\nu}+\Gamma^\lambda_{\mu\nu,\rho}-\Gamma^\eta_{\mu\rho}\Gamma^\lambda_{\nu\eta}+\Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\rho\eta},$$

$$\Gamma^\eta_{\mu\lambda}=\frac{1}{2}g^{\eta\nu}(g_{\mu\nu,\lambda}+g_{\lambda\nu,\mu}-g_{\mu\lambda,\nu}).$$

$$0=R_{(5)\mu\nu}-\frac{1}{2}g_{(5)\mu\nu}R-\frac{1}{2}\left(\frac{l^2}{12}+\Phi(\phi)\right)g_{(5)\mu\nu}$$
$$-\frac{1}{2}\left(\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{(5)\mu\nu}g^{\rho\sigma}_{(5)}\partial_\rho\phi\partial_\sigma\phi\right) \tag{18}$$

and from that over dilaton ϕ

$$0=\partial_\mu(\sqrt{g_{(5)}}g^{\mu\nu}_{(5)}\partial_\nu\phi)+\Phi'(\phi). \tag{19}$$

Assuming that $g_{(5)\mu\nu}$ is given by Eq. (17) and ϕ depends only on y: $\phi=\phi(y)$, we find that the equations of motion (18) and (19) take the following forms:

$$0=\frac{2kf}{y}-\frac{3}{2}\frac{1}{y^2}+\frac{1}{2}\left(\frac{l^2}{12}+\Phi(\phi)\right)f+\frac{1}{4}\left(\frac{d\phi}{dy}\right)^2, \tag{20}$$

$$0=\frac{kf}{y}+\frac{3}{4fy}\frac{df}{dy}+\frac{1}{2}\left(\frac{l^2}{12}+\Phi(\phi)\right)f-\frac{1}{4}\left(\frac{d\phi}{dy}\right)^2, \tag{21}$$

$$0=\frac{d}{dy}\left(\frac{y^2}{\sqrt{f}}\frac{d\phi}{dy}\right)+\Phi'(\phi)y^2\sqrt{f}. \tag{22}$$

Equation (20) corresponds to $(\mu,\nu)=(y,y)$ in Eq. (18) and Eq. (21) to $(\mu,\nu)=(i,j)$. The case of $(\mu,\nu)=(y,i)$ or (i,y) is identically satisfied.

On the other hand, on the (outer) brane, we obtain the following equations:

$$0=\frac{48l^4}{16\pi G}\left(\partial_z A-\frac{1}{l}-\frac{l}{24}\Phi(\phi)\right)e^{4A}+b'(4\partial_\sigma^4 A-16\partial_\sigma^2 A)$$
$$-4(b+b')[\partial_\sigma^4 A+2\partial_\sigma^2 A-6(\partial_\sigma A)^2\partial_\sigma^2 A]$$
$$+2C(\partial_\sigma^4\phi-4\partial_\sigma^2\phi), \tag{23}$$

$$0=-\frac{l^4}{8\pi G}e^{4A}\partial_z\phi-\frac{l^3}{32\pi G}e^{4A}\Phi'(\phi)$$
$$+C\{A(\partial_\sigma^4\phi-4\partial_\sigma^2\phi)+\partial_\sigma^4(A\phi)-4\partial_\sigma^2(A\phi)\}. \tag{24}$$

For the inner brane, one gets

$$0=-\frac{48l^4}{16\pi G}\left(\partial_z A-\frac{1}{l}-\frac{l}{24}\Phi(\phi)\right)e^{4A}+\tilde{b}'(4\partial_\sigma^4 A-16\partial_\sigma^2 A)$$
$$-4(\tilde{b}+\tilde{b}')[\partial_\sigma^4 A+2\partial_\sigma^2 A-6(\partial_\sigma A)^2\partial_\sigma^2 A]$$
$$+2\tilde{C}(\partial_\sigma^4\phi-4\partial_\sigma^2\phi), \tag{25}$$

$$0=\frac{l^4}{8\pi G}e^{4A}\partial_z\phi+\frac{l^3}{32\pi G}e^{4A}\Phi'(\phi)+\tilde{C}\{A(\partial_\sigma^4\phi-4\partial_\sigma^2\phi)$$
$$+\partial_\sigma^4(A\phi)-4\partial_\sigma^2(A\phi)\}. \tag{26}$$

In Eqs. (23) and (24), using the change of the coordinate $dz=\sqrt{f}dy$ and choosing $l^2e^{2\hat{A}(z,k)}=y(z)$ one uses the form of the metric as

$$ds^2=dz^2+e^{2A(z,\sigma)}\tilde{g}_{\mu\nu}dx^\mu dx^\nu,$$
$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu\equiv l^2(d\sigma^2+d\Omega_3^2). \tag{27}$$

Here $d\Omega_3^2$ corresponds to the metric of three-dimensional unit sphere. Then for the unit sphere $(k=3)$,

$$A(z,\sigma)=\hat{A}(z,k=3)-\ln\cosh\sigma, \tag{28}$$

for the flat Euclidean space $(k=0)$

$$A(z,\sigma)=\hat{A}(z,k=0)+\sigma, \tag{29}$$

and for the unit hyperboloid $(k=-3)$

$$A(z,\sigma)=\hat{A}(z,k=-3)-\ln\sinh\sigma. \tag{30}$$

We now identify A and $\tilde{g}$ in Eq. (27) with those in Eq. (7). Then we find $\tilde{F}=\tilde{G}=0$, $\tilde{R}=6/l^2$, etc.

Using Eqs. (20) and (22), one can delete f from the equations and can obtain an equation that contains only the dilaton field ϕ (and, of course, bulk potential):

$$0=\left\{\frac{5k}{2}-\frac{k}{4}y^2\left(\frac{d\phi}{dy}\right)^2+\left[\frac{3}{2}y-\frac{y^3}{6}\left(\frac{d\phi}{dy}\right)^2\right]\right.$$
$$\left.\times\left(\frac{6}{l^2}+\frac{1}{2}\Phi(\phi)\right)\right\}\frac{d\phi}{dy}+\frac{y^2}{2}\left(\frac{2k}{y}+\frac{6}{l^2}+\frac{1}{2}\Phi(\phi)\right)\frac{d^2\phi}{dy^2}$$
$$+\left[\frac{3}{4}-\frac{y^2}{8}\left(\frac{d\phi}{dy}\right)^2\right]\Phi'(\phi). \tag{31}$$

Our choice for dilaton and bulk potential admitting the analytical solution is

$$\phi(y)=p_1\ln(p_2y), \tag{32}$$

$$\Phi(\phi)=-\frac{12}{l^2}+c_1\exp(a\phi)+c_2\exp(2a\phi), \tag{33}$$

where a, p_1, p_2, c_1, and c_2 are some constants. When $p_1=\pm1/\sqrt{6}$, Eq. (31) is always satisfied but from Eq. (22), we find that $f(y)$ identically vanishes. Therefore we should assume $p_1\neq\pm1/\sqrt{6}$. Then we find the following set of exact bulk solutions:

$$\text{Case 1:}\quad c_1=\frac{6kp_2p_1^2}{3-2p_1^2},\quad c_2=0,\quad a=-\frac{1}{p_1},\quad p_1\neq\pm\sqrt{6}$$

$$f(y)=\frac{3-2p_1^2}{4ky} \tag{34}$$

$$\text{Case 2:}\quad c_1=-6kp_2,\quad a=\pm\frac{1}{\sqrt{3}},\quad p_1=\mp\sqrt{3}$$

$$f(y)=\frac{3}{(2c_2/p_2^2)-4ky} \tag{35}$$

Case 3: $c_2=3kp_2, \quad a=\pm\frac{1}{\sqrt{3}}, \quad p_1=\mp\sqrt{\frac{3}{2}}$

$$f(y)=\frac{21\sqrt{p_2}}{8\sqrt{y}(c_1y+7k\sqrt{p_2 y})}. \tag{36}$$

We can check that the above solutions satisfy Eq. (21).

In the coordinate system in Eq. (17), Eq. (24) for an outer brane has the following form:

$$0=-\frac{y_0^2}{8\pi G\sqrt{f(y_0)}}\partial_y\phi-\frac{y_0^2}{32\pi Gl}\Phi'(\phi_0)+6C\phi_0, \tag{37}$$

and Eq. (25) for the inner brane

$$0=\frac{\tilde{y}_0^{\,2}}{8\pi G\sqrt{f(\tilde{y}_0)}}\partial_y\phi+\frac{\tilde{y}_0^{\,2}}{32\pi Gl}\Phi'(\tilde{\phi}_0)+6\tilde{C}\tilde{\phi}_0. \tag{38}$$

Here ϕ_0 ($\tilde{\phi}_0$) is the value of the dilaton ϕ on the outer (inner) brane. We also find Eq. (23) for an outer brane has the following form:

$$0=\frac{3y_0^2}{16\pi G}\left(\frac{1}{2y_0\sqrt{f(y_0)}}-\frac{1}{l}-\frac{l}{24}\Phi(\phi_0)\right)+8b' \tag{39}$$

for $k\neq 0$ case and

$$0=\frac{3y_0^2}{16\pi G}\left(\frac{1}{2y_0\sqrt{f(y_0)}}-\frac{1}{l}-\frac{l}{24}\Phi(\phi_0)\right) \tag{40}$$

for $k=0$ case. For the inner brane (25) for $k\neq 0$ has the form of

$$0=-\frac{3\tilde{y}_0^{\,2}}{16\pi G}\left(\frac{1}{2\tilde{y}_0\sqrt{f(\tilde{y}_0)}}+\frac{1}{l}+\frac{l}{24}\Phi(\tilde{\phi}_0)\right)+8\tilde{b}'. \tag{41}$$

The equation for $k=0$ case is identical with that of the outer brane in Eq. (40) if we replace $\tilde{b}'$ with b'.

Case 1 solution

First we consider Case 1 in Eq. (34). Since $f(y)$ should be positive (we should also note $y>0$), one gets

$$q^2\equiv\frac{4k}{3-2p_1^2}>0, \quad q>0. \tag{42}$$

In Eq. (42), we can also consider the limit of $k\to 0$ by keeping q finite, i.e., $p_1^2\to\frac{3}{2}$.

When $k\neq 0$, Eqs. (39) and (37) have the following form:

$$\begin{aligned}-8b'=F_1(y_0)&\equiv\frac{3}{16\pi G}\left(\frac{q}{2}y_0^{3/2}-\frac{1}{2l}y_0^2-\frac{q^2p_1^2ly_0}{16}\right)\\ &=-\frac{3}{16\pi G}\frac{y_0}{2l}\left(y_0^{1/2}-\frac{1+\sqrt{1-\frac{p_1^2l^2}{2}}}{2}q\right)\\ &\quad\times\left(y_0^{1/2}-\frac{1-\sqrt{1-\frac{p_1^2l^2}{2}}}{2}q\right)\end{aligned} \tag{43}$$

$$0=-\frac{p_1q}{8\pi G}y_0^{3/2}+\frac{3lp_1q^2}{64\pi G}y_0+6C\phi_0, \tag{44}$$

and Eqs. (25) and (26) are

$$8\tilde{b}'=F_1(y_0)\equiv\frac{3}{16\pi G}\left(\frac{q}{2}y_0^{3/2}-\frac{1}{2l}y_0^2-\frac{q^2p_1^2ly_0}{16}\right) \tag{45}$$

$$0=\frac{p_1q}{8\pi G}y_0^{3/2}-\frac{3lp_1q^2}{64\pi G}y_0+6\tilde{C}\tilde{\phi}_0. \tag{46}$$

Since p_2 is absorbed into the definition of q in Eqs. (43) and (45), Eqs. (44) and (46) can be regarded as the equation that determines p_2 or $\phi_0/p_1\ln(p_2y_0)$ and $\tilde{\phi}_0/p_1=\ln(p_2\tilde{y}_0)$. We now investigate the properties of $F_1(y_0)$ as a function of y_0. The asymptotic behaviors are given by

$$F_1(y_0)\to-\frac{3}{16\pi G}\frac{p_1^2q^2l}{16}y_0<0 \quad \text{when } y_0\to+0 \tag{47}$$

$$F_1(y_0)\to-\frac{3}{16\pi G}\frac{1}{2l}y_0^2<0 \quad \text{when } y_0\to+\infty. \tag{48}$$

Since

$$F_1'(y_0)=\frac{3}{16\pi G}\left(\frac{3q}{4}y^{1/2}-\frac{1}{l}y_0-\frac{p_1^2q^2l}{16}\right), \tag{49}$$

$F_1(y_0)$ has extrema when

$$0=y_0-\frac{3ql}{4}y_0^{1/2}+\frac{p_1^2q^2l^2}{16}, \tag{50}$$

whose solutions are given by

$$y_0^{1/2}=y_\pm^{1/2}\equiv\frac{3ql}{8}\left(1\pm\sqrt{1-\frac{4p_1^2}{9}}\right). \tag{51}$$

Therefore if

$$|p_1|>\frac{3}{2}, \tag{52}$$

Eq. (50) does not have any solution and $F_1(y_0)$ is a monotonically decreasing function of y_0. Then Eqs. (47) and (48) tell that there is no solution of the brane equation (43) for negative b' in case of Eq. (52). On the other hand, when

$$|p_1|<\frac{3}{2}, \tag{53}$$

substituting Eq. (51) into the expression for $F_1(y_0)$ in Eq. (43), one gets

$$F_1(y_\pm)=\frac{3}{16\pi G}\frac{3^4q^4l^3}{2\times 8^4}\left(\sqrt{1-\frac{4p_1^2}{9}}\pm 1\right)\times\left(\sqrt{1-\frac{4p_1^2}{9}}\mp\frac{1}{3}\right). \tag{54}$$

Then we find $y_0=y_+$ corresponds to the maximum of $F_1(y_0)$. The maximum is positive $F_1(y_+)>0$ if $\sqrt{1-4p_1^2/9}-\frac{1}{3}>0$, that is,

$$p_1^2<2, \tag{55}$$

which is, of course, consistent with Eq. (53). In case of Eq. (55), if

$$F_1(y_+)\geq -8b', \tag{56}$$

Eq. (43) has a solution, that is, there can be a brane. We can also consider an inner brane that lies at $y=y_1<y_0$. For the inner brane, the relative sign of $\tilde{b}'$ and b' is changed in the equation corresponding to Eq. (43). Then if

$$F_1(y_-)\leq 8\tilde{b}', \tag{57}$$

there can be an inner brane. Then if both of Eqs. (56) and (57) hold, we can have a two-brane dilatonic solution. In case of two-brane solution, there might be, in general, a problem in the consistency between Eqs. (44) and (46). If we impose both of Eqs. (44) and (46), they can be regarded as the equations that determine p_1 and p_2 (we should note that p_2 is implicitly contained in ϕ_0 and $\tilde{\phi}_0$). In the classical limit, where $C=0$, there the terms containing p_2 (or ϕ_0 and $\tilde{\phi}_0$) disappear. Then it seems to be nontrivial if there exists any solution that satisfies both of Eqs. (56) and (57).

We now consider the classical limit in $k\neq 0$ case, where $b'=C=\tilde{b}'=\tilde{C}=0$ and Eqs. (43) and (45) become identical. Then the solutions of Eqs. (43) and (45) are given by

$$y_0^{1/2}=\left(1\pm\sqrt{1-\frac{p_1^2}{2}}\right)\frac{ql}{2}. \tag{58}$$

Since both of the solutions are positive, we can regard smaller one [$-$ sign in Eq. (58)] as expressing inner brane and larger one ($+$ sign) as an outer brane. On the other hand, Eqs. (44) and (46) have the following form:

$$0=\frac{p_1q^2ly_0}{2}\left(-\frac{1}{4}\mp\sqrt{1-\frac{p_1^2}{2}}\right). \tag{59}$$

In Eq. (59), the upper sign ($-$) corresponds to the outer brane and the lower one ($+$) to the inner brane. We should note that there is no solution, except for an outer brane $p_1^2=\frac{15}{8}$. This would tell that we need the quantum correction from brane matter in order to obtain the two-brane dilatonic inflationary universe where observable world may be associated with one of the inflationary branes.

When $k=0$ in Case 1, as discussed before, if q is finite, we find

$$p_1^2\to\frac{3}{2}. \tag{60}$$

Then Eq. (40) can be rewritten in the following form:

$$0=y_0-qly_0^{1/2}+\frac{3q^2l^2}{16}, \tag{61}$$

which has two solutions:

$$y_0^{1/2}=\frac{3ql}{4},\quad \frac{ql}{4}. \tag{62}$$

These two solutions might be regarded as two-brane solutions. On the other hand, the form of Eq. (37) for $k=0$ case is identical with that of $k\neq 0$ in Eq. (44), which can be solved with respect to ϕ_0 or p_2 in one-brane solution. However, in case $k=0$, the value of p_1 is fixed by Eq. (60). In the classical limit of $k=0$ case, Eq. (40) can be rewritten in the form of Eq. (61) and there appear solutions in Eq. (62). Equation (37) is, however, not satisfied. Equation (37) has a form of Eq. (59) but Eq. (60) does not satisfy Eq. (59). This demonstrates the role of quantum effects in realization of dilatonic inflationary two brane-world universe.

Case 2 solution

We now consider the case 2 in Eq. (35). Defining $\tilde{c}_2$ as

$$\tilde{c}_2\equiv\frac{c_2}{p_2^2}, \tag{63}$$

Eqs. (39) and (37) for the outer brane have the following form (when $k\neq 0$):

$$-8b'=F_2(y_0)\equiv\frac{3}{16\pi G}\left(\frac{y_0}{2}\sqrt{\frac{2\tilde{c}_2-4ky_0}{3}}-\frac{y_0^2}{2l}+\frac{kly_0}{4}-\frac{l\tilde{c}_2}{24}\right) \tag{64}$$

$$0=-\frac{y_0}{8\sqrt{3}\pi G}\sqrt{\frac{2\tilde{c}_2-4ky_0}{3}}-\frac{ly_0^2}{32\pi G}\times\left(-\frac{6k}{\sqrt{3}y_0}+\frac{2\tilde{c}_2}{\sqrt{3}y_0^2}\right)+6C\phi_0, \tag{65}$$

and Eqs. (41) and (38) for the inner brane, when $k\neq 0$:

$$8\tilde{b}'=F_2(\tilde{y}_0) \tag{66}$$

$$0=\frac{\tilde{y}_0}{8\sqrt{3}\pi G}\sqrt{\frac{2\tilde{c}_2-4k\tilde{y}_0}{3}}+\frac{l\tilde{y}_0^2}{32\pi G}\left(-\frac{6k}{\sqrt{3}\tilde{y}_0}+\frac{2\tilde{c}_2}{\sqrt{3}\tilde{y}_0^2}\right)+6\tilde{C}\tilde{\phi}_0. \tag{67}$$

Since p_2 is absorbed into the definition of $\tilde{c}_2$ in Eqs. (64) and (66), Eqs. (65) and (67) can be regarded again as the equation that determines p_2 or $\phi_0=p_1\ \ln(p_2y_0)$ $[\tilde{\phi}_0=p_1\ \ln(p_2\tilde{y}_0)]$. Then

$$F_2'(y_0)=\frac{3}{16\pi G}\left(\frac{1}{2}\sqrt{\frac{2\tilde{c}_2-4ky_0}{3}}-\frac{\frac{ky_0}{3}}{\sqrt{\frac{2\tilde{c}_2-4ky_0}{3}}}-\frac{y_0}{l}+\frac{kl}{4}\right). \tag{68}$$

Then if $F_2'(y_0)=0$, one gets

$$0=f(y_0)\equiv\frac{4k}{l^2}y_0^3+\left(k^2-\frac{2\tilde{c}_2}{l^2}\right)y_0^2+\left(\frac{k^3l^2}{4}-\tilde{c}_2k\right)y_0+\left(-\frac{k^2l^2}{8}+\frac{1}{3}\right)\tilde{c}_2 \tag{69}$$

and

$$f'(y_0)=\frac{12k}{l^2}y_0^2+2\left(k^2-\frac{2\tilde{c}_2}{l^2}\right)y_0+\left(\frac{k^3l^2}{4}-\tilde{c}_2k\right). \tag{70}$$

Then if we further put $f'(y_0)=0$, the determinant D of the equation is given by

$$\frac{D}{4}=\left(k^2-\frac{2\tilde{c}_2}{l^2}\right)^2-\frac{12k}{l^2}\left(\frac{k^3l^2}{4}-\tilde{c}_2k\right)=\frac{4}{l^2}\left\{\tilde{c}_2+k^2l^2\left(1+\sqrt{\frac{3}{2}}\right)\right\}\times\left\{\tilde{c}_2+k^2l^2\left(1-\sqrt{\frac{3}{2}}\right)\right\}. \tag{71}$$

If $D<0$, the equation $f'(y_0)=0$ does not have any solution. Then there can be only one solution $f(y_0)=0$, then in this case, $F_2(y_0)$ can have only one extremum. The explicit solutions of Eq. (69) are given by

$$y_0=-\frac{l^2k(1-2\hat{c})}{12k}+(-\beta+\sqrt{\beta^2+\alpha^3})^{1/3}\omega+(-\beta-\sqrt{\beta^2+\alpha^3})^{1/3}\omega^2$$

$$\omega=1,e^{2\pi i/3},e^{4\pi i/3}$$

$$\alpha\equiv\frac{2-8\hat{c}-4\hat{c}^2}{3}$$

$$\beta\equiv\frac{-7-12\hat{c}+96\hat{c}^2-16\hat{c}^3}{54}$$

$$\hat{c}=\frac{\tilde{c}_2}{k^2l^2}. \tag{72}$$

Then if

$$\beta^2+\alpha^3<0, \tag{73}$$

Eq. (69) has three different real solutions, and $F_2(y_0)$ can have three extrema (at maximum).

Let us consider the solution of Eq. (64) or the behavior of $F_2(y_0)$ in more detail. In case of $k>0$, since $F_2(y_0)$ contains $\sqrt{(2\tilde{c}_2-4ky_0)/3}$, the value of y_0 is restricted to $0\leqslant y_0\leqslant\tilde{c}_2/2k$ and $\tilde{c}_2$ should be positive: $\tilde{c}_2>0$. Since

$$F_2(0)=-\frac{3}{16\pi G}\frac{l\tilde{c}_2}{24}<0$$

$$F_2\left(\frac{\tilde{c}_2}{2k}\right)=-\frac{3}{16\pi G}\frac{1}{8k^2l}\left(\tilde{c}_2-\frac{2k^2l}{3}\right)\tilde{c}_2, \tag{74}$$

Eq. (64) has an outer-brane solution, at least, if $F_2(\tilde{c}_2/2k)\geqslant-8b'$. As usual $-8b'>0$, $F_2(\tilde{c}_2/2k)$ should be positive, which requires $\tilde{c}_2 < 2k^2 l/3$. More generally, since $F_2'(0) > 0$ and $F_2'(y_0 \to \tilde{c}_2/2k) \to -\infty$, $F_2(y_0)$ can have at least one maximum. If the maximum is greater than $-8b'$ ($=0$ in the classical case), there can be an outer-brane solution(s). And if $F_2(0) < 8\tilde{b}'$ ($=0$ in the classical case), there can be an inner-brane solution(s). We should note that such an inner and/or outer brane(s) solution(s) can exist in general even if $b' = \tilde{b}' = 0$. Hence, the possibility of creation of inflationary two-brane-world universe occurs not only on quantum but also on classical level (depending on the choice of the parameters).

We now consider the case of $k<0$ for Eq. (64) in the Case 2. If $\tilde{c}_2 > 0$, y_0 can take a value from 0 to positive infinity: $0 \leqslant y_0 < \infty$. Since

$$F_2(0) = -\frac{3}{16\pi G}\frac{l\tilde{c}_2}{24} < 0, \quad F_2'(0) = \frac{3}{16\pi G}\left(\sqrt{\frac{2\tilde{c}_2}{6}} + \frac{kl}{4}\right)$$

$$F_2(y_0 \to +\infty) \to -\frac{3}{16\pi G}\frac{y_0^2}{2l} < 0, \tag{75}$$

if $\tilde{c}_2 > 3k^2 l^2/8$, $F_2(y_0)$ has at least one maximum. If the value of the maximum is larger than $-8b'$ ($=0$ in the classical case), there is always an outer-brane solution. Even if $\tilde{c}_2 < 3k^2 l^2/8$, from Eq. (71), there can be a maximum when $\tilde{c}_2 > k^2 l^2 (\sqrt{\frac{3}{2}} - 1)$. Here we should note $\frac{3}{8} > \sqrt{\frac{3}{2}} - 1$. When $\tilde{c}_2 < k^2 l^2 (\sqrt{\frac{3}{2}} - 1)$, $F_2(y_0)$ becomes a monotonically decreasing function of y_0. Since $F_2(0)$ is negative, there cannot be any outer-brane solution.

In case that $k<0$ and $\tilde{c}_2 < 0$, we find $y_0 > \tilde{c}_2/2k$ in order that $f(y)$ is positive. The surface of $y_0 = \tilde{c}_2/2k$ can be regarded as a horizon. Since

$$F_2\left(\frac{\tilde{c}_2}{2k}\right) = -\frac{3}{16\pi G}\frac{1}{8k^2 l}\left(\tilde{c}_2 - \frac{2k^2 l}{3}\right)\tilde{c}_2 < 0$$

$$F_2(y_0 \to +\infty) \to -\frac{3}{16\pi G}\frac{y_0^2}{2l} < 0, \tag{76}$$

and $F_2(y_0 \to \tilde{c}_2/2k) \to +\infty$, there is at least one maximum. If the maximum is larger than $-8b'$ (when $b' < 0$), there are outer-brane solutions. And if $F_2(\tilde{c}_2/2k) < 8\tilde{b}'$, there can be an inner-brane solution.

Finally, when $k=0$ in Case 2 (35), the brane equation (40) has the following form:

$$0 = y_0^2 - l y_0 \sqrt{\frac{2\tilde{c}_2}{3}} + \frac{2l^2 \tilde{c}_2}{3}. \tag{77}$$

Then $\tilde{c}_2$ should be positive. The solution of Eq. (77) is given by

$$y_0 = \frac{l\sqrt{\tilde{c}_2}}{2}\left(\sqrt{\frac{2}{3}} \pm \frac{1}{\sqrt{3}}\right). \tag{78}$$

Since both of the two solutions are positive, there can be a solution with both of inner and outer branes.

Case 3 solution

We now briefly consider Case 3 (36). First, one should note that $p_2 > 0$ since the solution (36) contains $\sqrt{p_2}$. Then if we define $\tilde{c}_1$ as

$$\tilde{c}_1 \equiv \frac{c_1}{\sqrt{p_2}}, \tag{79}$$

when $k \neq 0$, Eqs. (39) and (37) have the following form:

$$-8b' = F_3(y_0)$$

$$\equiv \frac{3}{16\pi G}\left\{\frac{y_0}{2y_0}\sqrt{\frac{8\sqrt{y_0}(\tilde{c}_1 y_0 + 7k\sqrt{y_0})}{21}} - \frac{y_0^2}{2l} - \frac{l}{24}\sqrt{y_0}(\tilde{c}_1 y_0 + 3k\sqrt{y_0})\right\} \tag{80}$$

$$0 = \frac{y_0}{16\pi G}\frac{\sqrt{2\sqrt{y_0}(\tilde{c}_1 y_0 + 7k\sqrt{y_0})}}{7} - \frac{l\sqrt{3y_0}}{96\pi G}\left(3kp_2\tilde{c}_1 y_0 + \frac{2c_2\sqrt{y_0}}{p_2}\right) - 3\sqrt{3}C\phi_0. \tag{81}$$

Since p_2 is absorbed into the definition of $\tilde{c}_1$ in Eq. (80), Eq. (81) can be regarded as the equation that determines p_2 or $\phi_0 = p_1 \ln(p_2 y_0)$.

When $\tilde{c}_1$ is negative, we find $k>0$ and $0 < y_0 < 49k^2/\tilde{c}_1^2$ in order that $F_3(y_0)$ is real. Since

$$F_3(y_0 \to 0) = -\frac{3}{16\pi G}\frac{lky_0}{8} < 0$$

$$F_3\left(\frac{49k^2}{\tilde{c}_1^2}\right) = \frac{3}{16\pi G}\frac{(7k)^3}{\tilde{c}_1^2}\left(\frac{l}{42} - \frac{7k}{2l\tilde{c}_1^2}\right). \tag{82}$$

Then if $F_3(49k^2/\tilde{c}_1^2) > -8b'$ ($=0$ in the classical case), there can be an outer-brane solution.

When $\tilde{c}_1$ is positive, y_0 can take a value from 0 to $+\infty$ if k is positive. Since

$$F_3(y_0 \to 0) = -\frac{3}{16\pi G}\frac{lky_0}{8} < 0,$$

$$F_3(y_0 \to +\infty) = -\frac{3}{16\pi G}\frac{y_0^2}{2l} < 0, \tag{83}$$

it is not so clear if there can be any outer-brane solution.

When $\tilde{c}_1 > 0$ and $k<0$, we find $y_0 > 49k^2/\tilde{c}_1^2$ in order that $F_3(y_0)$ is real. Since

$$F_3\left(\frac{49k^2}{\tilde{c}_1^{\,2}}\right)=\frac{3}{16\pi G}\frac{(7k)^3}{\tilde{c}_1^{\,2}}\left(\frac{l}{42}-\frac{7k}{2l\tilde{c}_1^{\,2}}\right)>0$$

$$F_3(y_0\to+\infty)=-\frac{3}{16\pi G}\frac{y_0^2}{2l}<0, \tag{84}$$

there always exists an outer-brane solution if $F_3(49k^2/\tilde{c}_1^{\,2})>-8b'$.

We now summarize the obtained results. Generally the obtained bulk solutions have the form (the transformation of metric is discussed below)

$$\phi(y)=p_1\ln(p_2y)$$

$$\Phi(\phi)=-\frac{12}{l^2}+c_1\exp(a\phi)+c_2\exp(2a\phi). \tag{85}$$

(1) Case 1

(a) Bulk solution

$$c_1=\frac{6kp_2p_1^2}{3-2p_1^2},\quad c_2=0,\quad a=-\frac{1}{p_1},\quad p_1\neq\pm\sqrt{6}$$

$$f(y)=\frac{3-2p_1^2}{4ky}. \tag{86}$$

(b) When $k\neq0$ and $p_1^2<2$, there is an outer-brane solution if

$$F_1(y_+)\geqslant-8b', \tag{87}$$

and there is an inner-brane solution if

$$F_1(y_-)\leqslant 8\tilde{b}'. \tag{88}$$

Here F_1 is defined by

$$F_1(y_0)\equiv\frac{3}{16\pi G}\left(\frac{q}{2}y_0^{3/2}-\frac{1}{2l}y_0^2-\frac{q^2p_1^2ly_0}{16}\right) \tag{89}$$

and $y_\pm$ is given by

$$y_\pm^{1/2}\equiv\frac{3ql}{8}\left(1\pm\sqrt{1-\frac{4p_1^2}{9}}\right). \tag{90}$$

(c) Solution for $k=0$

$$p_1^2\to\frac{3}{2},\quad y_0^{1/2}=\frac{3ql}{4},\quad \frac{ql}{4}. \tag{91}$$

(2) Case 2

(a) Bulk solution

$$c_1=-6kp_2,\quad a=\pm\frac{1}{\sqrt{3}},\quad p_1=\mp\sqrt{3}$$

$$f(y)=\frac{3}{\frac{2c_2}{p_2^2}-4ky}. \tag{92}$$

(b) In case of $k>0$, $\tilde{c}_2\equiv c_2/p_2^2$ should be positive and there is an outer-brane solution, at least if $F_2(\tilde{c}_2/2k)\geqslant-8b'$, where

$$F_2(y_0)\equiv\frac{3}{16\pi G}\left(\frac{y_0}{2}\sqrt{\frac{2\tilde{c}_2-4ky_0}{3}}-\frac{y_0^2}{2l}+\frac{kly_0}{4}-\frac{l\tilde{c}_2}{24}\right). \tag{93}$$

(c) In case of $k<0$, $F_2(y)$ has at least one mimimum if $\tilde{c}_2<k^2l^2(\sqrt{3/2}-1)$ or $\tilde{c}_2>0$. If the value of $F_2(y)$ at the maximum is larger than $-8b'$, there is an outer-brane solution. If $\tilde{c}_2>0$ and $F_2(0)<8\tilde{b}'$ or $\tilde{c}_2<0$ and $F_2(\tilde{c}_2/2k)<8\tilde{b}'$, there can be an inner-brane solution.

(d) In case of $k=0$, if $\tilde{c}_2>0$, the solution is given by

$$y_0=\frac{l\sqrt{\tilde{c}_2}}{2}\left(\sqrt{\frac{2}{3}}\pm\frac{1}{\sqrt{3}}\right). \tag{94}$$

(3) Case 3

(a) Bulk solution

$$c_2=3kp_2,\quad a=\pm\frac{1}{\sqrt{3}},\quad p_1=\mp\frac{\sqrt{3}}{2}$$

$$f(y)=21\sqrt{\frac{p_2}{8\sqrt{y}}}(c_1y+7k\sqrt{p_2y}). \tag{95}$$

(b) When $\tilde{c}_1\equiv c_1/\sqrt{p_2}<0$, $k>0$, and there can be outer-brane solution if $F_3(49k^2/\tilde{c}_1^{\,2})>-8b'$, where

$$F_3(y_0)\equiv\frac{3}{16\pi G}\left\{\frac{y_0}{2y_0}\frac{\sqrt{8\sqrt{y_0}(\tilde{c}_1y_0+7k\sqrt{y_0})}}{21}-\frac{y_0^2}{2l}-\frac{l}{24}\sqrt{y_0}(\tilde{c}_1y_0+3k\sqrt{y_0})\right\}. \tag{96}$$

(c) When $\tilde{c}_1>0$ and $k<0$, there always exists outer-brane solution if $F_3(49k^2/\tilde{c}_1^{\,2})>-8b'$.

From the above results in cases (1)–(3), we find there very often appear two (inner and outer) branes solution as in the first model by Randall and Sundrum [1]. Moreover, the branes may be curved as de Sitter or hyperbolic space that gives the way for an ever-expanding inflationary universe. Such solutions often can exist even if there is no quantum effect, i.e., $b'=0$.

Let us make a few remarks on the form of the metric. If one considers the metric in the form (1), the warp factor $e^{2\tilde{A}(z)}$ does not behave as an exponential function of z but as a power of z. For example, in Case 1 (34),

$$ds^2=\frac{dy^2}{q^2y}+y\sum_{i,j=1}^{4}\hat{g}_{ij}dx^idx^j$$

$$=dz^2+\frac{z^2}{4q}\sum_{i,j=1}^{4}\hat{g}_{ij}dx^idx^j, \qquad (97)$$

where $z=2q\sqrt{y}$. In Case 2 (35)

$$ds^2=\frac{3dy^2}{2\tilde{c}-4ky}+y\sum_{i,j=1}^{4}\hat{g}_{ij}dx^idx^j$$

$$=dz^2+\left(\frac{\tilde{c}_2}{2}-\frac{2k^2z^2}{3k}\right)\sum_{i,j=1}^{4}\hat{g}_{ij}dx^idx^j, \qquad (98)$$

where $z=(1/k)\sqrt{3\tilde{c}_2/2-3ky}$. One should note that the exponential behavior of the warp factor $e^{A(z)}\sim e^{\gamma z}$ requires $f(y)\sim 1/y^2$, which tells that the space-time is nearly AdS:

$$ds^2\sim\frac{dy^2}{\gamma^2y^2}+y\sum_{i,j=1}^{4}\hat{g}_{ij}dx^idx^j$$

$$=dz^2+e^{\gamma z}\sum_{i,j=1}^{4}\hat{g}_{ij}dx^idx^j, \qquad (99)$$

where $y=e^{\gamma z}$. This would require that we need a region (of complete space-time) where, the potential and the dilaton become almost constant. It results in difficulties when one tries to explain the hierarchy using this model.

Hence, we presented a number of dilatonic (inflationary, flat or hyperbolic) two-brane-world universes that are created by quantum effects of brane matter. Sometimes, such universes may be realized due to specific choice of dilatonic potential even on a classical level.

IV. MULTI-BRANE GENERALIZATION

In some papers (for example, in [17]), the solution with many branes was proposed. In such a model, there are two AdS spaces with different radii or different values of the cosmological constants. They are glued by a brane, whose tension is given by the difference of the inverse of the radii. In the solution, the value of dA/dz in the metric of the form (1) jumps at the brane, which tells the value of $f(y)$ in the metric choice in Eq. (17) jumps on the brane since $\sqrt{f(y)}=dz/dy=1/(2ydA/dz)$. Imagine one includes the quantum effects on the brane. Then one can, in general, glue two AdS-like spaces with same values of the cosmological constant. Let us assume that there is a brane at $y=\hat{y}_0$ and there are two AdS-like spaces in $y>\hat{y}_0$ and $y<\hat{y}_0$ glued by the brane. One now denotes the quantity in the AdS-like space in $y>\hat{y}_0$ ($y<\hat{y}_0$) by the suffix $+$ $(-)$. If we consider the case where the value of l is identical in two AdS-like spaces and the value of the dilaton is continuous at the brane, we do not need the counterterm corresponding to S_1 in Eq. (5) or $\tilde{S}_1$ in Eq. (14) since the action corresponding to S_1 cancels with $\tilde{S}_1$ (note that the relative sign between S_1 and $\tilde{S}_1$ is opposite). Then instead of Eqs. (37) and (39) or (38) and (41), one obtains

$$0=-\frac{\hat{y}_0^2}{8\pi G}\left(\frac{\partial_y\phi_+(\hat{y}_0)}{\sqrt{f_+(\hat{y}_0)}}-\frac{\partial_y\phi_-(\hat{y}_0)}{\sqrt{f_-(\hat{y}_0)}}\right)+6\hat{C}\hat{\phi}_0(\hat{y}_0) \qquad (100)$$

$$0=\frac{3\hat{y}_0}{16\pi G}\left(\frac{1}{2\sqrt{f_+(\hat{y}_0)}}-\frac{1}{2\sqrt{f_-(\hat{y}_0)}}\right)+8\hat{b}' \qquad (101)$$

for $k\neq 0$ cases. Here we denote the quantities on the brane by $\hat{}$. For $k\neq 0$, we cannot put a brane without making the cosmological constants in the AdS-like spaces $y>\hat{y}_0$ and $y<\hat{y}_0$ different as in the case of [17] where the no quantum corrections case has been considered.

As an example, we only limit to Case 1 in Eq. (34)

$$f_\pm=\frac{1}{q_\pm^2y}. \qquad (102)$$

Then using Eq. (101), one finds

$$0=\frac{3\sqrt{\hat{y}_0}}{16\pi G}\left(\frac{1}{2q_+}-\frac{1}{2q_-}\right)+8\hat{b}'. \qquad (103)$$

On the other hand, from Eq. (100), we obtain

$$0=-\frac{\hat{y}_0^{3/2}}{8\pi G}(q_+p_{1+}-q_-p_{1-})+6\hat{C}\phi(\hat{y}_0). \qquad (104)$$

The condition of the continuity of the dilaton field at the brane gives, from Eq. (32),

$$\phi(\hat{y}_0)=p_{1+}\ \ln(p_{2+}\hat{y}_0)=p_{1-}\ \ln(p_{2-}\hat{y}_0). \qquad (105)$$

Equations (103), (104), and (105) are compatible with each other [note that $q_\pm$ is given in terms of $p_{1\pm}$ by Eq. (42)]. Let $\hat{y}_0$ and q_+ (or p_{1+}) be independent parameters. Then Eq. (103) can be solved with respect to q_-:

$$q_-=q_-(q_+,\hat{y}_0)\equiv\frac{1}{\frac{1}{q_+}+\frac{16\pi G\times 16\hat{b}'}{3\sqrt{\hat{y}_0}}}. \qquad (106)$$

Then putting $\phi(\hat{y}_0)=p_{1+}\ \ln(p_{2+}\hat{y}_0)$ in Eq. (104), we can solve the equation with respect to p_{2+}

$$p_{2+}=p_{2+}(q_+,\hat{y}_0)$$

$$\equiv\frac{1}{y_0}\exp\left(\frac{1}{6\hat{C}p_1(q_1)}\cdot\frac{\hat{y}_0^{3/2}}{8\pi G}\{q_+p_{1+}(q_+)-q_-(q_+,\hat{y}_0)p_{1-}[q_-(q_+,\hat{y}_0)]\}\right). \qquad (107)$$

Here $p_{1\pm}(q_\pm)$ is defined by solving Eq. (42):

$$p_{1\pm}(q_\pm)=\sqrt{\frac{3}{2}-\frac{2k}{q_\pm^2}}. \tag{108}$$

Finally, Eq. (105) can be solved with respect to p_{2-}:

$$p_{2-}=\frac{[p_{2+}(q_+,\hat{y}_0)\hat{y}_0]^{p_1(q_1)/\{p_{1-}[q_-(q_+,\hat{y}_0)]\}}}{y_0}. \tag{109}$$

When $k>0$, Eq. (108) gives a constraint

$$q_\pm^2>\frac{4k}{3}. \tag{110}$$

As long as the constraint in Eq. (110) holds, iterating the above procedure, we can obtain curved multi-brane solutions. Hence, we outlined the way to generalize the two-brane-world for a multi-brane case.

V. DISCUSSION

In summary, we presented the generalization of quantum dilatonic brane-world [10] where brane is flat, spherical (de Sitter), or hyperbolic and it is induced by quantum effects of CFT living on the brane. In this generalization one may have two brane-worlds or even multi-brane-worlds that proves the general character of the scenario suggested in Refs. [7,8] where instead of arbitrary brane tension added by hands the effective brane tension is produced by boundary quantum fields. What is more interesting, the bulk solutions have analytical form, at least, for specific choice of bulk potential under consideration.

In classical dilatonic gravity the variety of brane-world solutions has been presented in Ref. [18] where also the question of singularities has been discussed. The fine-tuned example of bulk potential where one gets a bulk solution that is not singular has been presented. Let us consider if our solutions contain the curvature singularity or not. Multiplying $g_{(5)}^{\mu\nu}$ with the Einstein equation in the bulk:

$$0=R_{(5)\mu\nu}-\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi-\frac{1}{2}g_{(5)\mu\nu}\times\left(R_{(5)}-\frac{1}{2}\nabla_\rho\phi\nabla^\rho\phi+\frac{12}{l^2}+\Phi(\phi)\right), \tag{111}$$

which is obtained from $S_{\rm EH}$ in Eq. (2), one gets

$$R_{(5)}=\frac{1}{2}\nabla_\rho\phi\nabla^\rho\phi-\frac{5}{3}\left(\frac{12}{l^2}+\Phi(\phi)\right). \tag{112}$$

Substituting expressions (32) and (33) into Eq. (112), we find

$$R_{(5)}=\frac{p_1^2}{2y^2f}-\frac{5}{3}[c_1(p_2y)^{ap_1}+c_2(p_2y)^{2ap_1}]. \tag{113}$$

Then for cases $1\sim3$, the scalar curvature $R_{(5)}$ is given by

$$\text{Case 1}\quad R_{(5)}=-\frac{3}{2}\frac{p_1^2q^2}{y} \tag{114}$$

$$\text{Case 2}\quad R_{(5)}=\frac{8k}{y}-\frac{2\tilde{c}_2}{3y^2} \tag{115}$$

$$\text{Case 3}\quad R_{(5)}=-\frac{33\tilde{c}_1}{21\sqrt{y}}-\frac{4k}{y}. \tag{116}$$

In all cases the singularity appears at $y=0$.

In Case 1, when $y\sim0$ and the coordinates besides y are fixed, the infinitesimally small distance ds is given by

$$ds=\sqrt{f}dy\sim\frac{dy}{q\sqrt{y}}, \tag{117}$$

which tells that the distance between the brane and the singularity is finite. Then in cases of $k=0$ and $k<0$, the singularity is naked when we Wick-rerotate space-time to Lorentzian signature. When $k>0$, the singularity is not exactly naked after the Wick-rerotation since the horizon is given by $y=0$, i.e., the horizon coincides with the curvature singularity.

In Case 2, the situation is not changed for $k=0$, $k>0$ and $k<0$ with $\tilde{c}_2>0$ from that in Case 1 and the distance between the brane and the singularity is finite since $ds\sim(dy/\sqrt{y})\sqrt{3/2\tilde{c}_2}$ when y is small. When $k<0$ with $\tilde{c}_2<0$, however, the singularity is not naked since there is a kind of horizon at $y=\tilde{c}_2/2k$, where $1/f(y)=0$. We should note the scalar curvature $R_{(5)}$ in Eq. (114) is finite. This tells that y is not a proper coordinate when $y\sim\tilde{c}_2/2k$. If a new coordinate η is introduced,

$$\eta^2\equiv2\left(y-\frac{\tilde{c}_2}{2k}\right), \tag{118}$$

the metric in Eq. (17) is rewritten as follows:

$$ds^2=-\frac{3}{4k}d\eta^2+\left(\frac{\tilde{c}_2}{2k}+\frac{\eta^2}{2}\right)\sum_{i,j=1}^{4}\hat{g}_{ij}(x^k)dx^idx^j. \tag{119}$$

The radius of 4D manifold with negative k, whose metric is given by $\hat{g}_{ij}$, has a minimum $\tilde{c}_2/2k$ at $\eta=0$, which corresponds to $y=\tilde{c}_2/2k$. The radius increases when $|\eta|$ increases. Therefore the space-time can be regarded as a kind of wormhole, where two universes corresponding to $\eta>0$ and $\eta<0$, respectively, are joined at $\eta=0$.

In Case 3, the singularity is naked (the singularity is not exactly naked when $k>0$ as in Case 1) in general and the distance between the brane and the horizon is finite except $k>0$ and $\tilde{c}_1<0$ case since there is a horizon at $\sqrt{y}=-7k/\tilde{c}_1$ where the scalar curvature (115) is finite.

The price for having analytical bulk results (exactly solvable bulk potential) is the presence of (naked) singularity. One can, of course, present the fine-tuned examples of bulk potential as in Refs. [12,18] where the problem of singularity does not appear. Moreover, bulk-quantum effects may significantly modify classical bulk configurations [8,19,20] that presumbly may help in the resolution of (naked) singularity problem. However, in such situations there are no analytical bulk solutions in dilatonic gravity.

There are various ways to extend the results of the present work. First of all, one can construct multi-brane dilatonic solutions within the current scenario for another class of bulk potentials. However, this requires the application of numerical methods. Second, it would be interesting to describe the details of brane-world-anomaly-driven inflation (with nontrivial dilaton) at late times when it should decay to standard Friedmann-Robertson-Walker (FRW) cosmology. Third, within similar scenarios one can consider dilatonic brane-world black holes that are currently under investigation.

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